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# Quantization of Yang–Mills Theory

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The canonical formulation of a constrained system is discussed. Quantization of the massive Yang–Mills field as an application of a field theory containing second-class constraints is studied. The set of Hamilton–Jacobi partial differential equations and the path integral of these theories are obtained by using the Muslih method.

## 1. INTRODUCTION

The generalized Hamiltonian dynamics describing systems with constraints was initiated by Dirac [1, 2] and is widely used in investigating theoretical models in contemporary elementary particle physics [3, 4]. The presence of constraints in such theories requires care when applying Dirac's method, especially when first-class constraints arise, since the first-class constraints are generators of gauge transformations which lead to the gauge freedom. Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: so-called first-class and second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first-class are by definition second-class.

Most physicists believe that this distinction is quite important not only in classical theories, but also in quantum mechanics [3, 4].

In the case of unconstrained systems, the Hamilton–Jacobi theory provides a bridge between classical and quantum mechanics. The first study of the Hamilton–Jacobi equations for arbitrary first-order actions was initiated

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by Santilli [5]. The quantization and construction of the functional integral for theories with first-class constraints in canonical gauge was given by Faddeev [6]. Faddeev's method is generalized by Senjanovic [7] to the case when second-class constraints appear in the theory. Moreover, Fradkin [8] considered quantization of bosonic theories with first- and second-class constraints and the extension to include fermions in such gauges. Gitman and Tyutin [3] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. Recently the Hamiltonian–Jacobi approach [9–11] has been developed to investigate constrained systems. The equivalent Lagrangian method [12] is used to obtain the set of Hamilton–Jacobi partial differential equations (HJPDE). In this approach the distinction between first- and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables, which require the investigation of integrability conditions. In other words, the integrability conditions may lead to new constraints. Moreover, it is shown that gauge fixing, which is an essential procedure to study singular systems by Dirac's method, is not necessary if the canonical method is used [11]. The path integral formulation based on the canonical method is obtained in refs. 13–15.

In this paper, we shall treat massive Yang–Mills theories as constrained systems. The path integral is obtained by using the Senjanovic and the canonical methods.

## 2. PATH INTEGRAL FORMULATION

In this section, we briefly review the Senjanovic method and the Hamilton–Jacobi method for studying the path integral for constrained systems.

### 2.1. Senjanovic Method

Consider a mechanical system with $m$ first-class constraints and $2n$ second-class constraints. Let the first-class constraints be called $\varphi_a$, the second-class constraints $\theta_a$, and the gauge conditions associated with the first-class constraints $\chi_a$. Let the $\chi_a$ be chosen in such a way that $\{\chi_a, \chi_b\} = 0$.

Then the expression for the $S$-matrix element is [7]

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left\{\int_{-\infty}^{+\infty} (p_i \dot{q}_i - H)\, dt\right\} \prod_t d\mu(q(t), p(t)) \qquad (1)$$

where $H$ is the Hamiltonian of the system and the measure of integration is defined by

$$d\mu(q, p) = \prod_a \delta(\chi_a)\, \delta(\varphi_a) \left|\det\left\|\{\chi_a, \varphi_a\}\right\|\right| \times \prod_c \delta(\theta_a) \left|\det\left\|\{\theta_a, \theta_b\}\right\|\right|^{1/2} \prod_i dp_i\, dq_i \tag{2}$$

## 2.2. Muslih Method

One starts from singular Lagrangian $L = L(q_l, \dot{q}_i, t)$, $i = 1, \ldots, n$, with the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \tag{3}$$

of rank $(n - r)$, $r < n$. The generalized momenta $p_i$ corresponding to the generalized coordinates $q_i$ are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, (n - r) \tag{4}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \qquad \mu = n - r + 1, \ldots, n \tag{5}$$

where $q_i$ are divided into two sets, $q_a$ and $x_\mu$. Since the rank of the Hessian matrix is $(n - r)$, one may solve Eq. (4) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_t, \dot{x}_\mu, p_a; t) \tag{6}$$

Substituting Eq. (6) into Eq. (5), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t) \tag{7}$$

The canonical Hamiltonian $H_0$ reads

$$H_0 = -L(q_i, \dot{x}_\nu, \dot{q}_a; t) + p_a \dot{q}_a - \dot{x}_\mu H_\mu, \qquad \nu = 1, 2, \ldots, r \tag{8}$$

The set of HJPDE is expressed as

$$H'_\alpha\left(x_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta}\right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, r \tag{9}$$

where

$$H'_0 = p_0 + H_0 \tag{10}$$

$$H'_\mu = p_\mu + H_\mu \tag{11}$$

We define $p_\beta = \partial S[q_a; x_a]/\partial x_\beta$ and $p_a = \partial S[q_a; x_a]/\partial q_a$ with $x_0 = t$ and $S$ being the action.

Now the total differential equations are given as

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a}\, dx_\alpha \tag{12}$$

$$dp_a = \frac{\partial H'_\alpha}{\partial q_a}\, dx_\alpha \tag{13}$$

$$dp_\beta = \frac{\partial H'_\alpha}{\partial t_\beta}\, dx_\alpha \tag{14}$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dx_\alpha \tag{15}$$

where $z = S(x_\alpha, q_a)$. These equations are integrable if and only if [11]

$$dH'_0 = 0 \tag{16}$$

$$dH'_\mu = 0, \qquad \mu = 1, 2, \ldots, r \tag{17}$$

If conditions (16), (17) are not satisfied identically, one considers them as new constraints and again consider their variations. Thus, repeating this procedure, one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the set of canonical phase space coordinates $(q_a, p_a)$ as functions of $t_a$; the canonical action integral is obtained in terms of the canonical coordinates. $H'_\alpha$ can be interpreted as the infinitesimal generator of canonical transformations given by parameters $t_\alpha$, respectively. In this case the path integral representation can be written as [13–15]

$$\langle \mathrm{out}|S|\mathrm{in}\rangle = \int \prod_{a=1}^{n-p} dq^a\, dp^a \exp\left\{\int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha\right\} \tag{18}$$

$$a = 1, \ldots, n-p, \qquad \alpha = 0, n-p+1, \ldots, n.$$

In fact, this path integral is an integration over the canonical phase-space coordinates $(q^a, p^a)$.

## 3. AN EXAMPLE

As an example, let us consider the Lagrangian density for the massive Yang–Mills theory as

$$\ell = -\frac{1}{4} F^\alpha_{\mu\nu}(x) F^{\mu\nu}_\alpha(x) + \frac{1}{2} M^2 A^\mu_\alpha(x) A^\alpha_\mu(x) \tag{19}$$

In Eq. (19), $F^{\mu\nu}_\alpha(x)$ is given by the formula

$$F_{\alpha}^{\mu\nu}(x) = \partial^{\mu} A_{\alpha}^{\nu}(x) - \partial^{\nu} A_{\alpha}^{\mu}(x) + g f_{\alpha\beta\gamma} A_{\beta}^{\mu}(x) A_{\gamma}^{\nu}(x) \tag{20}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of the Lie algebra and $g$ represents the coupling constant.

The momenta $\pi_{\alpha}^{i}$, $\pi_{\alpha}^{0}$ are defined as

$$\pi_{\alpha}^{i} = \frac{\partial \ell}{\partial(\dot{A}_{i}^{\alpha})} = -F_{\alpha}^{0i} \tag{21}$$

and

$$\pi_{\alpha}^{0} = \frac{\partial \ell}{\partial(\dot{A}_{0}^{\alpha})} = 0 \tag{22}$$

is the primary constraint.

Equation (21) leads us to express the velocities $\dot{A}_{i}^{\alpha}$ as

$$\dot{A}_{\alpha}^{i} = [\pi_{i}^{\alpha} - \partial_{i} A_{\alpha}^{0} + g f_{\alpha\beta\gamma} A_{\beta}^{0} A_{i}^{\gamma}] \tag{23}$$

The Hamiltonian density is given by

$$\begin{aligned} H = {} & \frac{1}{2} \pi_{i}^{\alpha} \pi_{i}^{\alpha} - \pi_{i}^{\alpha} \partial_{i} A_{0}^{\alpha} - g f^{\alpha\beta\gamma} \pi_{\alpha}^{i} A_{0}^{\beta} A_{i}^{\gamma} \\ & + \frac{1}{4} F_{\alpha}^{ik} F_{ik}^{\alpha} - \frac{1}{2} M^{2} A_{\alpha}^{02} + \frac{1}{2} M^{2} A_{\alpha}^{i2} \end{aligned} \tag{24}$$

Making use of (9)–(11), we find for the set of-HJPDE

$$H_{0}' = H_{0} + \pi_{4}^{\alpha} = 0 \tag{25}$$

$$H_{1}' = \pi_{0}^{\alpha} = 0 \tag{26}$$

where

$$\begin{aligned} H_{0} = \int \Bigg[ & \frac{1}{2} \pi_{i}^{\alpha} \pi_{i}^{\alpha} - \pi_{i}^{\alpha} \partial_{i} A_{0}^{\alpha} - g f^{\alpha\beta\gamma} \pi_{\alpha}^{i} A_{0}^{\beta} A_{i}^{\gamma} \\ & + \frac{1}{4} F_{\alpha}^{ik} F_{ik}^{\alpha} - \frac{1}{2} M^{2} A_{\alpha}^{02} + \frac{1}{2} M^{2} A_{\alpha}^{i2} \Bigg] d^{3}x \end{aligned} \tag{27}$$

The equations of motion are obtained as total differential equations follows:

$$\begin{aligned} dA_{\alpha}^{i}(x) &= \frac{\partial H_{0}'}{\partial \pi_{i}^{\alpha}} dt + \frac{\partial H_{1}'}{\partial \pi_{i}^{\alpha}} dA_{0}^{\beta} \\ &= [\pi_{i}^{\alpha}(x) - \partial_{i} A_{0}^{\alpha}(x) + g f^{\alpha\beta\gamma} A_{0}^{\beta}(x) A_{i}^{\gamma}(x)]\, dt \end{aligned} \tag{28}$$

$$d\pi^i_\alpha(x) = -\frac{\partial H'_0}{\partial A^\alpha_i}\,dt - \frac{\partial H'_1}{\partial A^\alpha_i}\,dA^\beta_0$$
$$= [gf^{\alpha\beta\gamma}\pi^i_\beta(x)A^\gamma_0(x) - M^2A^\alpha_i(x)$$
$$- \partial_k F^{ki}_\alpha(x) - F^{ik}_\gamma(x)gf^{\alpha\beta\gamma}A^\beta_k(x)]\,dt \tag{29}$$

$$d\pi^0_\alpha(x) = -\frac{\partial H'_0}{\partial A^\alpha_0}\,dt - \frac{\partial H'_1}{\partial A^\alpha_0}\,dA^\beta_0$$
$$= [\partial_i\pi^i_\alpha(x) - gf^{\alpha\beta\gamma}\pi^i_\beta(x)A^\gamma_i(x) + M^2A^0_\alpha(x)]\,dt = 0 \tag{30}$$

$$d\pi^\alpha_4(x) = -\frac{\partial H'_0}{\partial t}\,dt - \frac{\partial H'_1}{\partial t}\,dA^\beta_0 = 0 \tag{31}$$

To check whether the set of equations (28)–(31) is integrable or not, let us consider the total variation of Eq. (25). In fact,

$$dH'_0 = dH_0 + d\pi^\alpha_4 \tag{32}$$
$$= -(\partial_i\pi^i_\alpha - gf^{\alpha\beta\gamma}A^\gamma_i\pi^i_\beta + M^2A^0_\alpha)\,dA^\alpha_0 = 0 \tag{33}$$

Making use of (15), (25), and (33), we can write the canonical action integral as

$$z = \int \Bigg[\pi^i_\alpha A^\alpha_i - \frac{1}{2}\,\pi^i_\alpha\pi^i_\alpha - \frac{1}{2M^2}\,(\partial_i\pi^i_\alpha - gf^{\alpha\beta\gamma}A^\gamma_i\pi^i_\beta)$$
$$\times (\partial_n\pi^n_\alpha - gf^{\alpha\delta t}A^t_k\pi^k_\delta) - \frac{1}{2}\,M^2A^i_\alpha A^i_\alpha - \frac{1}{4}\,F^{kn}_\alpha F^\alpha_{kn}\Bigg]\,d^3x \tag{34}$$

Now the $S$-matrix element is given by

$$\langle \text{out}|S|\text{in}\rangle = \int dA^i_\alpha\,d\pi^i_\alpha \exp\left[i\int\left(-H_\alpha + p_a\,\frac{\partial H'_\alpha}{\partial p_a}\right)dt_\alpha\right]$$
$$= \int \prod_{i,\alpha} dA^i_\alpha\,d\pi^i_\alpha \exp i\int\Bigg[\pi^i_\alpha A^\alpha_i - \frac{1}{2}\,\pi^i_\alpha\pi^i_\alpha$$
$$- \frac{1}{2M^2}\,(\partial_i\pi^i_\alpha - gf^{\alpha\beta\gamma}A^\gamma_i\pi^i_\beta)$$
$$\times (\partial_n\pi^n_\alpha - gf^{\alpha\delta t}A^t_k\pi^k_\delta) - \frac{1}{2}\,M^2A^i_\alpha A^i_\alpha - \frac{1}{4}\,F^{kn}_\alpha F^\alpha_{kn}\Bigg]\,d^4x \tag{35}$$

Now we will apply the Senjanovic method to the previous example. The total Hamiltonian is given as

$$H_T = \int \left( \frac{1}{2} \pi_i^\alpha \pi_i^\alpha - \pi_i^\alpha \partial_i A_0^\alpha - g f^{\alpha\beta\gamma} \pi_\alpha^i A_0^\beta A_i^\gamma + \frac{1}{4} F_\alpha^{ik} F_{ik}^\alpha - \frac{1}{2} M^2 A_\alpha^{02} + \frac{1}{2} M^2 A_\alpha^{i2} \right) d^3x + \lambda_\alpha \pi_\alpha^0 \tag{36}$$

where $\lambda_\alpha$ is a Lagrange multiplier to be determined. Imposing the consistency condition $\dot{\pi}_\alpha^0 = 0$ leads to

$$\{\pi_\alpha^0, H_T\} = \partial_i \pi_\alpha^i - g f^{\alpha\beta\gamma} \pi_\beta^i A_i^\gamma + M^2 A_\alpha^0 \tag{37}$$

as a secondary constraint. Imposing the consistency condition

$$\{\partial_i \pi_\alpha^i - g f^{\alpha\beta\gamma} \pi_\beta^i A_i^\gamma + M^2 A_\alpha^0, H_T\} = 0 \tag{38}$$

we arrive at the result

$$M^2 \lambda_\alpha + \{\partial_i \pi_\alpha^i - g f^{\alpha\beta\gamma} \pi_\beta^i A_i^\gamma, H_0\} = 0 \tag{39}$$

which determinates $\lambda_\alpha$ and no further constraints arise.

Making use of (18), we obtain

$$\begin{aligned} \langle \mathrm{out}|S|\mathrm{in}\rangle = \int \prod_{\alpha,i} DA_i^\alpha D\pi_\alpha^i \prod_\alpha D\pi_\alpha^0 DA_\alpha^0 \det(M^2 I) \\ \times \prod_\alpha \{\delta(\pi_\alpha^0)\, \delta(\partial_i \pi_\alpha^i - g f^{\alpha\beta\gamma} A_i^\gamma \pi_\beta^i + M^2 A_\alpha^0)\} \\ \times \exp\left\{ i \int \left[ \pi_0^\alpha \dot{A}_0^\alpha + \pi_\alpha^i \dot{A}_i^\alpha - \frac{1}{2} \pi_\alpha^i \pi_\alpha^i \right.\right. \\ - \pi_\alpha^i (\partial_n A_\alpha^0 - g f^{\alpha\delta t} A_0^\beta A_i^\gamma) + \frac{1}{2} M^2 A_\alpha^\mu A_\mu^\alpha \\ \left.\left. - \frac{1}{4} F_{lm}^\alpha F_\alpha^{lm} \right] d^4x \right\} \end{aligned} \tag{40}$$

Integrating over $A_\alpha^0$, and $\pi_\alpha^0$ one can arrive at the result (35).

## 4. CONCLUSION

Path integral quantization of a massive Yang–Mills field is obtained by using the canonical path integral formulation [13–15], since the integrability conditions $dH_0'$ and $dH_1'$ are satisfied. This system is integrable, hence the path integral is obtained directly as an integration over the canonical phase space coordinates $(\pi^i_\alpha, A^i_\alpha)$. In the usual formulation [7] one has to integrate over the extended phase space $(\pi^0_\alpha, A^0_\alpha, \pi^i_\alpha, A^i_\alpha)$ and one can get rid of the redundant variables $(\pi^0_\alpha, A^0_\alpha)$ by using delta functions $\delta(\pi^0_\alpha)$ and $\delta(\partial_i \pi^i_\alpha - qf^{\alpha\beta\gamma}A^\gamma_i\pi^i_\beta + M^2A^0_\alpha)$.

As a conclusion, the Muslih method is simpler and more economical there is no need to distinguish between first and secondary constraints, and there is no need to introduce Lagrange multipliers; all that is needed is the set of Hamilton–Jacobi partial differential equations and the equations of motion. If the system is integrable then one can construct the canonical phase space.

One should notice that for $f^{\alpha\beta\gamma} = 0$ ($\alpha$ takes a single value) the results obtained here reduce to the case for a massive vector field in electromagnetic theory [15]. The results for $M \to 0$ reduce to the case for a free field in electromagnetic theory [14].

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